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C^1 Approximations of Inertial Manifolds for Dissipative Nonlinear Equations

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In this paper we study a class of nonlinear dissipative partial differential equations that have inertial manifolds. This means that the long-time behavior is equivalent to a certain finite system of ordinary differential equations. We investigate ways in which these finite systems can be approximated in the C^1 sense. Geometrically this may be interpreted as constructing manifolds in phase space that are C^1 close to the inertial manifold of the partial differential equation. Under such approximations the invariant hyperbolic sets of the global attractor persist.

1. Introduction

Since one can rarely write down the analytical solutions to nonlinear dissipative partial differential equations (PDEs), it is important to understand whether, and in what sense, the behavior of approximating numerical schemes to these equations reflects the true dynamics of the original equations. Further, in the case of interesting dynamics standard error estimates between approximations and the true solutions coming from spectral methods, finite difference or finite element schemes for example, grow exponentially in time due to instabilities. Hence, in this case standard error analysis provides little value in understanding the long-time behavior of a given approximating scheme.

In this paper we will not add to the understanding of how the dynamics of general nonlinear dissipative PDEs behaves under approximation. Rather, we will consider PDEs whose long-time behavior is equivalent to that of an finite dimensional ordinary differential equation (ODE). We may

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then apply the existing theory of approximating dynamics for ODEs to find appropriate approximations to the PDEs which preserve essential parts of their long-time dynamics. For example, consider the ODE

$$\frac{dx}{dt} = X(x), \tag{1.1}$$

where we suppose that $x \in \mathbb{R}^n$ and that X is a C^1 vector field. Further, we suppose that the system (1.1) is dissipative, and hence has a global attractor. Now suppose that (1.1) is approximated by

$$\frac{dy}{dt} = X(y) + Y(y), \tag{1.2}$$

where $y \in \mathbb{R}^n$ and Y is a C^1 vector field, satisfying

$$\|Y\|_{C^1(\mathscr{U})} := \sup_{y \in \mathscr{U}} [|Y(y)| + \|DY(y)\|_{L(\mathbb{R}^n)}] \leqslant \varepsilon$$

in an appropriate neighborhood $\mathscr{U}$ of the global attractor for some suitable $\varepsilon > 0$. That is, (1.1) and (1.2) may be viewed as small C^1 perturbations of one another. This seems to be a natural condition to require of a perturbation in order to say something about how the global attractor of (1.2) relates to that of (1.1) and vise versa. It is known, for example, that normally hyperbolic invariant sets persist under such perturbations. Indeed, such systems have been studied by several authors and increasingly stronger results have been obtained, see for example [14], [30], [48], [50].

In order to apply the ODE results directly to PDEs, one must first construct finite systems of ODEs that have the same global attractor as that of the infinite-dimensional PDE. This has been done for several dissipative PDEs including, for example, the Kuramoto–Sivashinsky equation (interfacial instabilities, wrinkled flame fronts. etc...) [6, 7, 20, 21, 22], Cahn–Hilliard equation, (phase transitions) [6, 45], Ginzburg–Landau (hydrodynamic instabilities) [12, 26], and certain reaction-diffusion equations [6, 31, 43, 42]. Such systems are called *inertial forms.*

To be more specific, each of these equations can be viewed as an infinite-dimensional ordinary differential equation on a suitably chosen Hilbert space, H. We denote by $(\cdot, \cdot)$ the inner product and $|\cdot|$ the norm on H. Then these equations take the form

$$\begin{gathered} \frac{du}{dt} + Au + R(u) = f \\ u(0) = u_0, \end{gathered} \tag{1.3}$$

where $f \in H$ and the assumptions on $R(u)$ will be given below. For simplicity the linear operator A is assumed to be an unbounded positive self-adjoint operator with compact inverse (for the more general non-self-adjoint case see [51]). Thus there exists an orthonormal basis $\{\varphi_j\}$ of H consisting of eigenvectors of A,

$$A\varphi_j = \lambda_j \varphi_j,$$

where the λ_j satisfy $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_j \to \infty$ as $j \to \infty$ (for the details of this set up see, e.g., [6, 23, 53]).

In all cases the existence of inertial forms (IF) has been proven by showing the existence of an inertial manifold. To date inertial manifolds have been constructed as a graph in phase space of a Lipschitz function Φ (see [21], [22], and also see [2], [6], [7], [15], [19], [23], [37], [42]). An inertial manifold (IM) for a dissipative evolution partial differential equation is a smooth finite-dimensional manifold in phase space, which is positively invariant under the solution operator, and which uniformly attracts every bounded subset of phase space at an exponential rate. It is clear that if the IM exists then it must contain the global attractor. Moreover, the reduction of the partial differential equation to the IM yields the inertial form.

Even in the case that the IM or a smooth function Φ does not exist the theory suggests to look for a global function Φ_{app} whose graph in phase space approximates the attractor. The projection of the PDE onto the manifold Graph(Φ_{app}) yields a finite system suitable for computations. This is the major idea behind the theory of *approximate inertial manifolds* and the associated *nonlinear Galerkin methods*. (see [10], [11], [15], [16], [17], [18], [23], [24], [33], [34], [36], [38], [43], [44], [54], [57], [56]). We emphasize that the goal of these efforts is not necessarily to produce a scheme that gives more accurate results on a finite interval of time (see however [11] and the references therein). Rather it is to produce a finite system that accurately reflects the long-time dynamics of the original PDE.

To date most existence proofs of IMs have required the linear operator A to have large gaps in its spectrum—see condition (2.5) below (see, however, [2] for an exception). We assume that A satisfies this gap condition throughout this paper.

We denote by P the orthogonal projection of the space H onto $PH = \text{span}\{\varphi_1, ..., \varphi_M\}$ and $Q = I - P$. We set $p = Pu$, $q = Qu$. Then the evolution equation (1.3) is equivalent to the system

$$\frac{dp}{dt} + Ap + PR(p+q) = Pf,$$

$$\frac{dq}{dt} + Aq + QR(p+q) = Qf.$$

If the IM is given as a graph of a Lipschitz function $\Phi: PH \mapsto QD(A)$, then on this manifold the solutions of (1.3) are of the form $u(t) = p(t) + \Phi(p(t))$. Moreover, in this case the inertial form is given by

$$\frac{dp}{dt} + Ap + PR(p + \Phi(p)) = Pf \quad p \in PH. \tag{1.4}$$

The main goal of this method of reduction is to implement the reduced ordinary differential system (1.4) in long-time simulations of solutions to the PDE (1.3). For this purpose one would need to know the explicit form of the function Φ. However, this is not available except in very special cases, as in [2]. Therefore, one would have to provide an appropriate approximate function Φ_{app}, and instead use the corresponding approximate inertial form

$$\frac{dp}{dt} + Ap + PR(p + \Phi_{app}(p)) = Pf \quad p \in PH, \tag{1.5}$$

in computations. Let us remark that sometimes one should use a variant of the approximate inertial form (1.5) in order to preserve the global dissipative nature of the PDE (1.3) (for details see [34]). If one chooses Φ_{app} to be close to Φ in the C^1 norm, that is,

$$\sup_{p \in PH} (|A(\Phi_{app}(p) - \Phi(p))| + \|A(D\Phi_{app}(p) - D\Phi(p))\|_{L(PH,\, QH)}) \leqslant \varepsilon,$$

where $D\Phi$ denotes the Fréchet derivative of the function Φ, then the vector field in the approximate inertial form (1.5) may be viewed as a small C^1 perturbation of the vector field in the inertial form (1.4). In this case one can say something about how the dynamics of the approximate inertial form (1.5) relates to the dynamics of the IF (1.4), and hence, to the dynamics of the PDE (1.3) itself. Indeed, one may apply the results of [14], [48], for example, to conclude that certain compact overflowing, inflowing, invariant (normally hyperbolic) manifolds persist under such perturbations (more general structures are considered in [48]). Stable stationary and periodic orbits are example of such invariants sets. Thus we expect results like [5], [55, 56, 58] (which are proven for a Galerkin system approximating the Navier–Stokes equations) to be a consequence of the C^1 closeness of (1.4) and (1.5) for the equations under study here.

To carry on with this argument, first one has to make sure that the IM is a graph of a C^1 function. Let us recall that under the same spectral gap condition (2.5), which guarantees the existence of a Lipschitz IM, Φ, one can show that the function Φ is indeed C^1. In fact, one can also show that the larger the gaps in the spectrum of the linear operator A the smoother is function Φ (for the details of these statements see [3], [8], [42], [46]).

Based on the above discussion, the main objective of this work is to introduce a new method for approximating the IM arbitrarily closely in the C^1-norm. The method utilized here also shows that the inertial forms associated with finite differences, finite elements, and spectral methods approximating the PDEs mentioned above can be made C^1 close to the inertial form of the PDE (see [35] for the finite difference case). That the spectral method based on the eigenfunctions of the linear operator A has an IM and inertial form is studied in [22], [23]. More general approximating schemes are studied in [9], [25], [37], [39].

We organize the paper as follows. In Section 2 we give the specific assumptions on the nonlinear term $R(u)$. These conditions are satisfied by all the physical equations mentioned above. Then we prove a slightly stronger version of the strong squeezing property, which was first introduced in [19, 20] for the Kuramoto–Sivashinsky equation and later developed for a general abstract equation in [23]. This result is the essential tool used throughout this paper. In Section 3 we present an infinite-dimensional damped hyperbolic PDE on the space PH with values in $QD(A)$. This PDE has a unique stationary solution which is asymptotically stable. This stationary solution is an inertial manifold for the equation (1.3). In Section 4 we show that the solutions to this infinite-dimensional PDE converge exponentially fast to the IM, with an exponential rate proportional to λ_{M+1}, where M is the dimension of the IM. As an application, one can take any of the currently available AIM, the ones studied in [17], [23], [43], [54], [57], which are already close to the IM, as an initial value for the infinite-dimensional PDE and integrate it forward for a very short interval of time to get an even closer AIM, in the C^1 norm. In Section 5 we conclude by showing that a Galerkin approximation of Equation (1.3) is a small C^1 perturbation of (1.4), the inertial form, provided the system is taken sufficiently large. Thus one may deduce that certain structures of the global attractor for the infinite-dimensional PDE are preserved by this Galerkin scheme (see also [32]). Hence, we may apply our convergence results to the Galerkin system, which is finite-dimensional system.

2. Preliminary Results

We begin with the specific assumptions about the nonlinear term $R(u)$. We assume (as in [23]) $R(u)$ to be a differentiable map from $\mathscr{D}(A)$ into $\mathscr{D}(A^{1-\beta})$ for some β, $0 \leqslant \beta \leqslant 1/2$ and satisfies the following inequalities:

$$|R'(u)v| \leqslant M_0(|Au|)\,|A^{\beta}v|,$$
$$|A^{1-\beta}R'(u)v| \leqslant M_1(|Au|)\,|Av|$$

for all $u, v \in \mathscr{D}(A)$. $M_k: \mathbb{R}^+ \mapsto \mathbb{R}^+$, $k=1, 2$, are given monotonic increasing functions.

We suppose that for every $u_0 \in \mathscr{D}(A)$, Equation (1.3) has a unique global solution which will be denoted $S(t)u_0$. In addition, this solution satisfies $S(t)u_0 \in \mathscr{D}(A)$ for all $t \geqslant 0$ and is continuous in both variables. Since we are only assuming that $f \in H$, in general one can not assume more regularity of the solutions. That the system (1.3) is dissipative means there exists a ball of radius ρ_0 in $\mathscr{D}(A)$, $B(0, \rho_0)$, that is absorbing; that is, for every $r > 0$ there exists a $T(r) \geqslant 0$ such that $S(t)\, B(0, r) \subset B(0, \rho_0)$ for all $t \geqslant T$ (see [28], [53]). Thus, due to the absorbing property, we may truncate the nonlinear term outside the ball $B(0, 2\rho_0)$. This will avoid certain technical difficulties at infinity in $\mathscr{D}(A)$. However, the resulting modified equation will provide the same dynamics inside the absorbing ball $B(0, \rho_0)$. In particular, it will provide the same asymptotic behavior as $t \to \infty$ near the global attractor.

Let $\theta: \mathbb{R}^+ \mapsto [0, 1]$ be a fixed smooth function with $\theta(s) = 1$ for $0 \leqslant s \leqslant 2$, $\theta(s) = 0$ for $s \geqslant 4$, and $|\theta'(s)| \leqslant 2$ for $s \geqslant 0$. Fix $M_2 = 2\rho_0$ and define $\theta_{M_2}(s) = \theta(s/M_2^2)$. The modified equation of (1.3) is

$$\frac{du}{dt} + Au + F(u) = f, \tag{2.1}$$

where $F(u) = \theta_{M_2}(|Au|^2)\, R(u)$. We also assume the cut-off function θ muliplies the forcing term, f, though it is not explicit written. From now on the solution $u(t)$ of Equation (2.1) will also be denoted by $S(t)u_0$. We now recall the following proposition from [23].

Proposition 2.1. *There exists constants K_1, K_2 that only depend on β, M_1 such that the nonlinear operator $F(u)$ satisfies the estimates*

$$|A^{1-\beta}F(u)| \leqslant K_1 \qquad \forall u \in \mathscr{D}(A) \tag{2.2}$$

$$|A^{1-\beta}F'(u)v| \leqslant K_2\, |Av| \qquad \forall u, v \in \mathscr{D}(A) \tag{2.3}$$

$$|A^{1-\beta}(F(u_1) - F(u_2))| \leqslant K_2\, |A(u_1 - u_2)| \quad \forall u_1, u_2 \in \mathscr{D}(A). \tag{2.4}$$

We remark that so far the conditions in this section hold for a wide class of dissipative equations, including the $2\mathscr{D}$ Navier–Stokes equations, Kuramoto–Sivashinsky equation, non-local Burgers equation, Cahn–Hilliard equation, complex Ginzburg–Landau equation, and certain nonlinear reaction diffusion equations. We will now, however, restrict ourself by requiring the spectrum of the operator A to have large gaps. This condition, among other things, will guarantee the existence of an inertial manifold. Unfortunately, the Navier–Stokes equations are unlikely to satisfy this condition.

As before, we denote by P the orthogonal projection onto the first M eigenvectors of the linear operator A, and let $Q=I-P$. Let $\gamma>0$ be given. We denote by $\Gamma_M(\gamma)$ the cone in $\mathscr{D}(A)\times\mathscr{D}(A)$ defined by

$$\Gamma_M(\gamma)=\{[u_1,u_2]: u_1,\ u_2\in\mathscr{D}(A),\ |QA(u_1-u_2)|\leqslant\gamma\ |PA(u_1-u_2)|\}.$$

The next theorem is generally referred to as the strong squeezing property. It was first introduced for the Kuramoto–Sivashinsky equations in [19], [20], an abstract version of it was developed in [23]. Here, we prove a slightly stronger version than the one given in [23].

THEOREM 2.2. *Let u_1, u_2 be two solutions of* (2.1) *satisfying $|Au_i|\leqslant M_2$, for $i=1, 2$, and all $t\geqslant 0$, and with $f\in H$. Suppose that M can be chosen large enough so that $\lambda_{M+1}^{1-\beta}>2K_2(1+\gamma^{-1})$ and*

$$\lambda_{M+1}-\lambda_M>K_2((1+\gamma)\,\lambda_M^\beta+(1+\gamma^{-1})\,\lambda_{M+1}^\beta). \tag{2.5}$$

Then the following two statements hold:

(i) (uniform cone condition) *If $[u_1(t_0), u_2(t_0)]\in\Gamma_M(\gamma)$ for some $t_0\geqslant 0$, then $[u_1(t), u_2(t)]\in\Gamma_M(\gamma)$ for all $t\geqslant t_0$.*

(ii) (Strong squeezing property) *If $[u_1(t), u_2(t)]\notin\Gamma_M(\gamma)$ for all $0\leqslant t\leqslant T$, then*

$$|QA(u_1(t)-u_2(t))|\leqslant|QA(u_1(0)-u_2(0))|\ e^{-\lambda_{M+1}t/2}$$

for all $0\leqslant t\leqslant T$.

Proof. Denote by $u_{i,n}$ the solution of the Galerkin approximating system for (2.1)

$$\frac{du_{i,n}}{dt}+Au_{i,n}+P_nF(u_{i,n})=P_n f, \tag{2.6}$$

with initial data $u_{i,n}(t_0)=P_nu_i(t_0)$, for $i=1, 2$ and for $n>M$, where M is chosen as above. It is clear that the initial data satisfy $|A(u_{i,n}(t_0)-u_i(t_0))|\to 0$ as $n\to\infty$, for $i=1, 2$.

Set $V_n(t)=|AQ(u_{1,n}(t)-u_{2,n}(t))|-\gamma\,|AP(u_{1,n}(t)-u_{2,n}(t))|$. We first show that if $u_{1,n}$ and $u_{2,n}$ are two different solutions of (2.6) and if $V_n(t_1)=0$, for some $t_1\geqslant t_0$, then $dV_n(t_1)/dt<0$.

Setting $\varDelta=Q(u_1(t)-u_2(t))$, $\delta=P(u_1(t)-u_2(t))$, $\varDelta_n=Q(u_{1,n}(t)-u_{2,n}(t))$ and $\delta_n=P(u_{1,n}(t)-u_{2,n}(t))$ we get from (2.6)

$$\frac{d}{dt}\varDelta_n+A\,\varDelta_n+R_n(F(u_{1,n})-F(u_{2,n}))=0, \tag{2.7}$$

where R_n is the orthogonal projection onto the span$\{\varphi_{M+1}, ..., \varphi_n\}$. Taking the inner product of (2.7) with $A^2 \Delta_n$ gives (this is the basic reason for considering the Galerkin procedure (2.6), because we do not know a priori whether the difference $(u_1(t) - u_2(t))$ is in $\mathscr{D}(A^2)$ or not):

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A\,\Delta_n|^2 + |A^{3/2}\,\Delta_n|^2 &\leqslant |(R_n A^{1-\beta}(F(u_{1,n}) - F(u_{2,n})), A^{1+\beta}\,\Delta_n)|\\ &\leqslant K_2\,|A(u_{1,n} - u_{2,n})|\,\lambda_{M+1}^{\beta-1/2}\,|A^{3/2}\,\Delta_n|\\ &\leqslant K_2(|A\,\Delta_n| + |A\,\delta_n|)\,\lambda_{M+1}^{\beta-1/2}\,|A^{3/2}\,\Delta_n|,\end{aligned}$$

here we have used (2.4) and the fact that $0 \leqslant \beta \leqslant 1/2$.

Since $V_n(t_1) = 0$, then at $t = t_1$ we have

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}|A\,\Delta_n|^2 + |A^{3/2}\,\Delta_n|^2 &\leqslant K_2(\gamma^{-1}+1)|A\,\Delta_n|\,\lambda_{M+1}^{\beta-1/2}\,|A^{3/2}\,\Delta_n|\\ &\leqslant \frac{K_2(\gamma^{-1}+1)}{\lambda_{M+1}^{1-\beta}}\,|A^{3/2}\,\Delta_n|^2.\end{aligned}$$

Because of our choice for M (see (2.5)) we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}|A\,\Delta_n|^2 + \left(1 - \frac{K_2(\gamma^{-1}+1)}{\lambda_{M+1}^{1-\beta}}\right)|A^{3/2}\,\Delta_n|^2 \leqslant 0,\\ &\frac{d}{dt}|A\,\Delta_n| + \lambda_{M+1}\left(1 - \frac{K_2(\gamma^{-1}+1)}{\lambda_{M+1}^{1-\beta}}\right)|A\,\Delta_n| \leqslant 0.\end{aligned} \tag{2.8}$$

A similar analysis for the δ_n term yields, at $t = t_1$

$$\frac{1}{2}\frac{d}{dt}|A\,\delta_n|^2 + |A^{3/2}\,\delta_n|^2 \geqslant -K_2(\gamma+1)\,|A\,\delta_n|\,|A^{1+\beta}\,\delta_n|,$$

or

$$\frac{d}{dt}|A\,\delta_n| + \lambda_M|A\,\delta_n| \geqslant -K_2(1+\gamma)\,\lambda_M^{\beta}\,|A\,\delta_n|. \tag{2.9}$$

Subtracting (2.9) from (2.8) we get at $t = t_1$

$$\begin{aligned}&\frac{d}{dt}(|A\,\Delta_n| - \gamma\,|A\,\delta_n|)\\ &\qquad\leqslant [(\lambda_M - \lambda_{M+1}) + K_2((\gamma+1)\,\lambda_M^{\beta} + (\gamma^{-1}+1)\,\lambda_{M+1}^{\beta})]\,|A\,\Delta_n|.\end{aligned}$$

Thanks to (2.5) we have

$$\frac{dV_n(t_1)}{dt} = \frac{d}{dt}(|A\,\Delta_n| - \gamma\,|A\,\delta_n|) < 0.$$

We conclude from this that if $V_n(t_0) \leqslant 0$, then $V_n(t) \leqslant 0$ for all $t \geqslant t_0$.

Let $\tau > t_0$ be given. From (2.2), (2.3), (2.4) and (2.6) one can easily derive the following energy estimates

$$\sup_{t_0 \leqslant t \leqslant \tau} |Au_{i,n}(t)| \leqslant C_1$$

$$\int_{t_0}^{\tau} |A^{3/2}\,u_{i,n}(t)|^2\,dt \leqslant C_2$$

$$\int_{t_0}^{\tau} \left|A^{1/2}\,\frac{d}{dt}\,u_{i,n}(t)\right|^2\,dt \leqslant C_3,$$

where C_j, for $j = 1, 2, 3$, depend only on τ and $|Au_i(t_0)|$.

By applying the Aubin's compactness theorem, and by following similar arguments to the ones given in [4], [40] or [52] for the Navier–Stokes equations, one concludes that there is a subsequence $n_k \to \infty$ such that:

$u_{i,n_k} \to u_i$ weakly in $L^2(0, \tau; D(A^{3/2}))$

$u_{i,n_k} \to u_i$ strongly in $L^2(0, \tau; D(A))$

$u_{i,n_k} \to u_i$ strongly in $L^\infty(0, \tau; D(A^{1/2}))$

$u_{i,n_k} \to u_i$ weakly in $D(A)$ pointwise in $[t_0, \tau]$.

Because of the uniqueness of the solutions, the above limits hold for the whole sequence as well.

Since we are assuming $|AQ(u_1(t_0) - u_2(t_0))| \leqslant \gamma\,|AP(u_1(t_0) - u_2(t_0))|$, we have for all $n > M$

$$|A\,\Delta_n(t_0)| \leqslant |A\,\Delta(t_0)| \leqslant \gamma\,|A\,\delta(t_0)| = \gamma\,|A\,\delta_n(t_0)|,$$

hence, $V_n(t_0) \leqslant 0$, and consequently $V_n(t) \leqslant 0$ for all $t_0 \leqslant t \leqslant \tau$, and all $n > M$. In particular, we have $|A\,\Delta_{n_k}(t)| - \gamma\,|A\,\delta_{n_k}(t)| \leqslant 0$, for al $t_0 \leqslant t \leqslant \tau$, and all $n_k > M$.

Since $|A\,\Delta_{n_k}(t)|$ converges weakly in $\mathscr{D}(A)$, pointwise in $[t_0, \tau]$, we have

$$|A\,\Delta(t)| \leqslant \liminf_{n_k \to \infty} |A\,\Delta_{n_k}(t)| \leqslant \limsup_{n_k \to \infty} \gamma\,|A\,\delta_{n_k}(t)| = \gamma\,|A\,\delta(t)|,$$

for all $t_0 \leqslant t \leqslant \tau$. For the limit in the right hand side we have used the fact that the Galerkin solutions coverge strongly in $L^\infty(0, \tau; D(A^{1/2}))$, and since $\delta(t)$ and $\delta_{n_k}(t)$ lie in the finite-dimensional space PH, the convergence is also in the $D(A)$ norm. This concludes our proof for part (i).

To prove (ii) we again consider the solutions $u_{i,n}$ to the Galerkin projection (2.6) with initial data $u_{i,n}(0) = P_n u_i(0)$ and for $n > M$, $i = 1, 2$. Since $|A\, \Delta(0)| > \gamma\, |A\, \delta(0)|$, for sufficiently large n we have

$$|A\, \Delta_n(0)| > \gamma\, |A\, \delta_n(0)|.$$

Since the solutions of (2.6) are continuous in t, for each n large enough there exists a $T_n > 0$ such that $|A\, \Delta_n(t)| > \gamma\, |A\, \delta_n(t)|$ for all $0 \leqslant t \leqslant T_n$. Denote by

$$\bar{T}_n = \sup\{T_n \colon |A\, \Delta_n(t)| > \gamma\, |A\, \delta_n(t)| \text{ for all } t \in [0, T_n]\}.$$

Equation (2.8) holds on the interval $[0, \bar{T}_n)$, and we get from the Gronwall's inequality

$$|A\, \Delta_n(t)| \leqslant |A\, \Delta(0)|\, e^{-\lambda_{M+1} t/2} \qquad 0 \leqslant t < \bar{T}_n,$$

where we have required M so large that $1 - K_2(\gamma^{-1} + 1)\lambda_{M+1}^{\beta - 1} \geqslant 1/2$.

Now we claim that $T \leqslant \bar{T} = \liminf_{n \to \infty} \bar{T}_n$. Suppose not, that is assume $T > \bar{T}$. Then there exists a subsequence n_k such that $T > \bar{T}_{n_k}$. Therefore, we have $|A\, \Delta_{n_k}(T)| \leqslant \gamma\, |A\, \delta_{n_k}(T)|$. Then, as before we can show that

$$|A\, \Delta(T)| \leqslant \liminf_{n_k \to \infty} |A\, \Delta_{n_k}(T)| \leqslant \gamma \limsup_{n_k \to \infty} |A\, \delta_{n_k}(T)| = \gamma\, |A\, \delta(T)|.$$

This implies that $[u_1(T), u_2(T)] \in \Gamma_M(\gamma)$, which contradicts the assumptions in (ii). From this and (2) we conclude for n large enough we have

$$|A\, \Delta_n(t)| \leqslant |A\, \Delta(0)|\, e^{-\lambda_{M+1} t/2} \qquad 0 \leqslant t \leqslant T \leqslant T_n.$$

Taking the limit infimum again, we conclude that

$$|A\, \Delta(t)| \leqslant \liminf_{n \to \infty} |A\, \Delta_n(t)| \leqslant |A\, \Delta(0)|\, e^{-\lambda_{M+1} t/2}.$$

for all $0 \leqslant t \leqslant T$. ∎

Remark. The above proof can be simplified under the additional assumption that $f \in \mathscr{D}(A^{1-\beta})$. For in this case one can show that $du/dt \in \mathscr{D}(A^{1-\beta})$ and that $u(t) \in \mathscr{D}(A^{2-\beta})$ for all $t > 0$. In this case it is not necessary to first prove the cone condition; i.e. the strong squeezing property, for the Galerkin system and then pass to the limit (see [49]).

The existence of an inertial manifold as a graph of a function Φ now follows from the Hadamard approach as in [42] or [19], [20]. Furthermore, Φ enjoys the property

$$|A(\Phi(p_1)-\Phi(p_2))| \leqslant \gamma\, |A(p_1-p_2)|. \tag{2.10}$$

We will suppose throughout that M is sufficiently large to satisfy (2.5) so that the conclusions of Theorem 2.2 hold. It follows from our assumptions on $F(u)$ that the solution $u(t)=S(t)u_0$ of Equation (2.1) is differentiable with respect to the initial data (see for example [29], [53]). That is, the Fréchet derivative

$$\frac{DS(t)u_0}{Du_0} := S'(t, u_0)$$

exists and $S'(t, u_0)\mu_0 := \mu(t)$ is the solution of

$$\begin{aligned} &\frac{d\mu}{dt} + A\mu + F'(S(t)\,u_0)\mu = 0 \\ &\qquad\qquad\qquad\qquad \mu(0) = \mu_0. \end{aligned} \tag{2.11}$$

The following result is an immediate consequence of Theorem 2.2.

COROLLARY 2.3. If $|QA\mu(0)| \leqslant \gamma\, |PA\mu(0)|$, *then* $|QA\mu(t)| \leqslant \gamma\, |PA\mu(t)|$ *for all* $t \geqslant 0$.

3. THE PDE

We know that from [3], [8], [42], [46] that under the assumptions of Theorem 2.2 Φ is C^1 in $\mathscr{D}(A)$. Set $p=Pu$, $q=Qu$, where $PH=\operatorname{span}\{\varphi_1, ..., \varphi_M\}$ with M chosen so that (2.5) holds. Then by applying the projections P and Q to (2.1), we obtain the system

$$\frac{dp}{dt} + Ap + PF(p(t)+q(t)) = 0 \tag{3.1}$$

$$\frac{dq}{dt} + Aq + QF(p(t)+q(t)) = 0, \tag{3.2}$$

where for convenience we include the forcing term, f, in the term F. We still require only that $f \in H$. For any solution on the IM we have

$$q(t) = \Phi(p(t)), \tag{3.3}$$

where $p(t)$, $q(t)$ solve Equations (3.1), (3.2) respectively. Differentiating (3.3) gives

$$\frac{dq}{dt} = D\Phi \frac{dp}{dt}$$

or using Equations (3.1), (3.2)

$$A\Phi - D\Phi(p)(Ap + PF(p + \Phi(p))) + QF(p + \Phi(p)) = 0, \tag{3.4}$$

where the support of Φ is inside the ball $B(0, 4\rho_0)$. This restriction is, in a sense, a boundary condition for equation (3.4). We recall that conversely, if Φ is a solution of (3.4), then $\mathscr{M} = \mathrm{Graph}(\Phi)$ is an invariant manifold for Equation (2.1).

We wish to approximate solutions of (3.4). Our method for accomplishing this is partially motivated by the work of [15]. There an elliptic regularized version of Equation (3.4) is studied, namely,

$$-\varepsilon\, \Delta\Phi_\varepsilon(p) + A\Phi_\varepsilon(p) - D\Phi_\varepsilon(p)(Ap + PF(p + \Phi_\varepsilon(p))) + QF(p + \Phi_\varepsilon(p)) = 0, \tag{3.5}$$

where Δ is the Laplace operator on PH with zero Dirichlet boundary condition on the boundary of the ball $PB(0, 4\rho_0)$, and $\varepsilon > 0$ denotes an artificial viscosity. It is shown in [15] that $\Phi_\varepsilon \to \Phi$ in appropriate spaces as $\varepsilon \to 0$ (in fact, the existence of Φ is shown by considering Equation (3.5) as $\varepsilon \to 0$).

We instead consider the infinite-dimensional damped hyperbolic system

$$\begin{aligned} &\frac{\partial\Psi(\tau, p)}{\partial\tau} + A\Psi(\tau, p) - D\Psi(\tau, p)(Ap + PF(p + \Psi(\tau, p))) \\ &\qquad + QF(p + \Psi(\tau, p)) = 0 \\ &\Psi(0, p) = \Psi_0(p) \end{aligned} \tag{3.6}$$

where Ψ_0 is assumed to be a C^1 map from PH into $Q\mathscr{D}(A)$ such that

$$\sup_{p \in \mathscr{H}} \|AD\Psi_0(p)\|_{L(PH, QH)} \leqslant \gamma, \tag{3.7}$$

Also we suppose that $\mathrm{supp}(\Psi_0) \subset B(0, 4\rho_0)$.

The condition (3.7) is, in a sense, a smallness condition on the initial data in the C^1 norm. Such a condition should be expected because the equation (3.6) is a hyperbolic system, and one would not expect the damping term $A\Psi$ to be able to control sharp gradients. But it will prevent a shock from occurring if we start with “small” gradients.

Notice that Φ is a stationary solution of (3.6). It will be seen that Equation (3.6) is already sufficiently damped (even without a term like $-\varepsilon\Delta\Psi$ added) and that Ψ converges at an exponential rate to the stationary solution Φ which is an IM for (2.1), provided that Ψ_0 satisfies (3.7).

To show that (3.6) has a unique global solution for initial values satisfying (3.7) we use the method of characteristics. This will turn out to involve the Hadamard approach for the existence of the IM [42]. In particular, we will need the gap condition to prevent "shocks" from developing. More precisely, we will show that the function constructed using the Hadamard approach in [42] is the solution to (3.6) subject to the initial value $\Psi_0 \equiv 0$. Hence there will be some similarities between the techniques implemented by [42] in their geometric Hadamard approach and our analytic approach studying the infinite dimensional damped hyperbolic system (3.6).

The next lemma will show that $S(t)(PH)$ converges to Graph(Φ) as $t \to \infty$ (as in [23], [42]). The idea behind the approximate inertial manifold called Euler–Galerkin, which was introduced in [23], was to follow the evolution of the linear manifold PH ($\Psi_0 = 0$) under the semigroup $S(t)$ forward for a short time. Here, the hope is that by starting Equation (3.6) with initial data Ψ_0 that is already close enough to the IM the solution of Equation (3.6) will converge at an exponential rate to the IM. If this is so, one can improve (in the C^1 norm) any given approximation to the IM by integrating (3.6) forward for a short time.

Remark. One could also consider adding $-\varepsilon\,\Delta\Psi$ to the left hand side of equation (3.6) to obtain an infinite-dimensional parabolic system whose unique steady state is stable and is the solution to (3.5). Under certain smallness conditions on the initial data one would expect global existence of solutions to this parabolic system. Therefore, under the same smallness condition on the initial data one concludes that for ε sufficiently small the solutions to this parabolic system are close to the IM. This is because of the stability of the steady solution of the parabolic system and because for small ε this steady state solution of Equation (3.5) is close to the IM Φ. Moreover, these solutions are C^1 due to the regularizing effect of $-\varepsilon\,\Delta\Psi$ even though the initial value Ψ_0 may not be.

We therefore start by setting, for $t \geqslant 0$, $\mathcal{M}_t = S(t)(\mathrm{Graph}(\Psi_0))$ (rather than $S(t)(PH)$ as in [23], [42]). A weaker version of Lemma 3.8 and Lemma 3.2 below has been proven in [42]. The difference here is that we prove our results in a stronger norm, namely in the $\mathcal{D}(A)$ norm, by following the ideas of [42] and using our stronger version of the cone condition.

LEMMA 3.1. *For each $\tau \geqslant 0$ there exists a function $\chi(\tau, p)$: $PH \mapsto Q\mathcal{D}(A)$ such that*

$$M_\tau = \mathrm{Graph}(\chi(\tau, \cdot)),$$
$$|A(\chi(\tau, p_1) - \chi(\tau, p_2))| \leqslant \gamma |A(p_1 - p_2)| \tag{3.8}$$

for all $p_1, p_2 \in PH$, *and* $\chi(0, p) = \Psi_0$, *where* Ψ_0 *satisfies* (3.7).

We will show below that $\chi(\tau, p)$ is the solution of Equation (3.6).

Proof. We define the map as follows. For every $\tau_0 > 0$, $p_0 \in PH$ we have $p_0 = P(S(\tau_0)(p_1 + \Psi_0(p_1)))$ for some $p_1 \in PH$. That such a p_1 exists can be shown as follows: set $g(p) = -PS(\tau_0)(p + \Psi_0(p)) + p_0$. Then g: $PH \to PH$ is continuous. In addition, for p sufficiently large, $|Ap| \geqslant 2e^{\lambda_M \tau_0} \max\{|Ap_0|, 4p_0\}$, we have

$$\begin{aligned}(g(p), p) &= -(P(S(\tau_0)p), p) + (p, p_0) = -(e^{-PA\tau_0/2}p, e^{-PA\tau_0/2}p) + (p, p_0)\\ &\leqslant -|e^{-PA\tau_0/2}p|^2 + |p|\,|p_0| \leqslant -e^{-\lambda_M \tau_0}\,|p|^2 + |p|\,|p_0|.\end{aligned}$$

Hence,

$$\lim_{|p| \to \infty} \frac{(g(p), p)}{|p|} = -\infty.$$

Therefore, $(g(p), p) < 0$ on the boundary of a sufficiently large ball, hence by Lemma 7.2 p. 58 of [4] $g(p)$ has at least one zero inside this large ball.
Set

$$\chi(\tau_0, p_0) = Q(S(\tau_0)(p_1 + \Psi_0(p_1))).$$

χ is also well-defined. If there exists p_1 and p_2 such that

$$P(S(\tau_0)(p_1 + \Psi_0(p_1))) = P(S(\tau_0)(p_2 + \Psi_0(p_2))),$$

then, setting $u_i(t) = S(t)(p_i + \Psi_0(p_i))$ for $i = 1, 2$, it follows from the assumptions on Ψ_0 that at $t = 0$

$$\begin{aligned}|QA(u_1(0) - u_2(0))| &= |A(\Psi_0(p_1) - \Psi_0(p_2))|\\ &\leqslant \gamma\,|PA(p_1 - p_2)| = \gamma\,|PA(u_1(0) - u_2(0))|.\end{aligned}$$

From the uniform cone condition, Theorem 2.2, this last relation holds for all $t \geqslant 0$. In particular, at $t = \tau_0$. It follows that $u_1(\tau_0) = u_2(\tau_0)$, and from the backward uniqueness (see for example [1]) $u_1(t) = u_2(t)$ for all $t \geqslant 0$ and $p_1 = p_2$. Equation(3.8) is an immediate consequence of the cone condition and the definition of χ. ∎

It is important to notice that the only requirement used in Lemma 3.1 on Ψ_0 is the inequality $|A(\Psi_0(p_1) - \Psi_0(p_2))| \leqslant \gamma\,|A(p_1 - p_2)|$. In fact the argument in Lemma 3.1 can be applied to any function (graph in phase space) with this property. We will make repeated use of this fact throughout this paper.

The characteristic equations for (3.6) are

$$\frac{d\tau}{dt} = 1,$$

$$\frac{dp}{dt} = -Ap - PF(p + \psi) \tag{3.9}$$

$$\frac{d\psi}{dt} = -A\psi - QF(p + \psi), \tag{3.10}$$

where $\tau = \tau(t, p_0)$, $p = p(t, p_0)$ and $\psi = \psi(t, p_0)$, and with initial values $\tau(0, p_0) = 0$, $p(0, p_0) = p_0$, and $\psi(0, p_0) = \Psi_0(p_0)$, for arbitrary $p_0 \in PH$. After solving these equations one can implement the Implicit Function Theorem to invert the functions $\tau(t, p_0)$ and $p(t, p_0)$ to obtain $t = t(\tau, p)$, $p_0 = p_0(\tau, p)$ and substitute them in ψ to get $\Psi(\tau, p) = \psi(t(\tau, p), p_0(\tau, p))$ the solution of (3.6). The solutions of these characteristic equations are exactly the trajectories of (2.1), $S(t)u_0$, with initial data u_0 on Graph(Ψ_0). On the other hand, for any trajectory of the form $u(\tau) = S(\tau)(p_0 + \Psi_0(p_0))$ it follows from the definition of χ in Lemma 3.1 that

$$\chi(\tau, Pu(\tau)) = Qu(\tau).$$

Hence, whenever $\Psi(\tau, p)$ is defined by the characteristics method as above one concludes that $\chi(\tau, p) = \Psi(\tau, p)$.

The function $\chi(\tau, p)$ is globally defined for all $\tau \geq 0$. Hence, it is enough to show that $\chi(\tau, p)$ is a classical global solution of (3.6) which is equivalent to showing, by using the Implicit Function Theorem, the invertibility of the functions $\tau = \tau(t, p_0)$, $p = p(t, p_0)$ since $t(\tau, p_0) = t$. To achieve this one has to study the invertibility of the operator $\rho_{op} := \partial p(t, p_0)/\partial p_0$. For this we must consider the linearization of the system (3.9), (3.10)

$$\begin{aligned} &\frac{d\rho_{op}}{dt} + A\rho_{op} + PDF(u)(\rho_{op} + \sigma_{op}) = 0 \\ &\frac{d\sigma_{op}}{dt} + A\sigma_{op} + QDF(u)(\rho_{op} + \sigma_{op}) = 0, \end{aligned} \tag{3.11}$$

where $\sigma_{op} := \partial\psi(t, p_0)/\partial p_0$, $u(t, p_0) = p(t, p_0) + \psi(t, p_0)$ and $p(t, p_0)$, $\psi(t, p_0)$ solve the system (3.9), (3.10); equivalently, $u(t, p_0)$ is a solution of (2.1) with initial data on Graph(Ψ_0). That is, $u(0, p_0) = p_0 + \Psi_0(p_0)$. Notice that $\rho_{op}\colon PH \mapsto PH$ and $\sigma_{op}\colon PH \mapsto QH$ are linear operators.

THEOREM 3.2. *Let Ψ_0 satisfy* (3.7). *Then Equation* (3.6) *has a solution $\Psi(\tau, p)$ which is global in τ and p. Moreover, its partial Fréchet derivative with respect to p satisfies*

$$\|A(D\Psi(\tau, p))\|_{L(PH,\, QH)} \leqslant \gamma \tag{3.12}$$

for all $\tau \geqslant 0$.

Proof. Based on the above discussion let us observe that the characteristic system (3.9), (3.10) and it linearization (3.11) have global existence for all $t \geqslant 0$ and $p_0 \in PH$. To establish the theorem we need first to show that $\rho_{op}(t, p_0)$ is invertible for every t and every $p_0 \in PH$, which will guarantee the global invertibility of the function $p(t, p_0)$. From the initial condition $p(0, p_0) = p_0$, we have that $\rho_{op}(0, p_0) = P$ which is the identity on PH and this assures the invertibility of $p(t, p_0)$ for all $p_0 \in PH$ and $0 \leqslant t \leqslant T(p_0)$ for some $T(p_0) > 0$. Assume by contradiction that for some $t_0 > 0$ and $p_0 \in PH$ the matrix $\rho_{op}(t_0, p_0)$ is singular. Therefore, there exist a $\xi \in PH$, $\xi \neq 0$ such that $\rho_{op}(t_0, p_0)\xi = 0$.

Set $\rho(t, p_0) = \rho_{op}(t, p_0)\xi$, $\sigma(t, p_0) = \sigma_{op}(t, p_0)\xi$ and $\mu(t, p_0) = \rho(t, p_0) + \sigma(t, p_0)$. Notice that $\mu(t, p_0)$ satisfies (2.11). Because the initial condition Ψ_0 satisfies (3.7), we have that $|A\sigma_{op}(0)\xi| = |AD\Psi_0(p_0)\xi| \leqslant \gamma\,|A\xi| = \gamma\,|A\rho_{op}(0)\xi|$. Therefore, by Corollary 2.3, we obtain $|A\sigma(t, p_0)| \leqslant \gamma\,|A\rho(t, p_0)|$ for all $t \geqslant 0$. Since $\rho_{op}(t_0, p_0)\xi = 0$, the cone condition implies that $\sigma_{op}(t_0, p_0)\xi = 0$ and hence, $\mu(t_0, p_0) = 0$. Now the backward uniqueness (see [1]) implies that $\mu(t) = 0$ for all $0 \leqslant t \leqslant t_0$. However, $0 = P\mu(0) = \rho_{op}(0, p_0)\xi = I\xi$. Thus $\xi = 0$ which is a contradiction. As a result $\Psi(\tau, p)$ exists globally and is differentiable. Moreover, since $\Psi = \chi$ and χ satisfies (3.8), (3.12) follows. ∎

4. CONVERGENCE RESULTS

To show that the solution Ψ of Equation (3.6) converges to Φ is a simple consequence of the strong squeezing property. Let $p \in PH$ and $\tau > 0$ be given. From Lemma 3.1 there exists $u_{0a} \in \mathrm{Graph}(\Psi_0)$, $u_{0m} \in \mathrm{Graph}(\Phi)$ such that $p = PS(\tau)\, u_{0a} = PS(\tau)\, u_{0m}$. Further, $\Psi(\tau, p) = QS(\tau)\, u_{0a}$, $\Phi(p) = QS(\tau)\, u_{0m}$. We will denote throughout by using the subscript m solutions on the inertial manifold and we will denote with the subscript a solutions starting on $\mathrm{Graph}(\Psi)$. Since the solutions $u_a = S(t)\, u_{0a}$, $u_m = S(t)\, u_{0m}$ of Equation (2.1) are not in the cone $\Gamma_M(\gamma)$ for all $0 \leqslant t \leqslant \tau$, Theorem 2.2 (ii) applies and

$$|QA(u_a(t) - u_m(t))| \leqslant e^{-\lambda_{M+1} t/2}\, |QA(u_{0a} - u_{0m})| \tag{4.1}$$

for this interval of time. Since Ψ_0, Φ are uniformly bounded, we have at $t=\tau$

$$\sup_{p\in PH} |A(\Psi(\tau,p)-\Phi(p))| \leqslant 2K_3 e^{-\lambda_{M+1}\tau/2} \quad \tau \geqslant 0, \tag{4.2}$$

where $K_3 = \max_{p\in PH}\{|A\Psi_0(p)|, |A\Phi(p)|\}$.

We would like to improve the estimate given in (4.2). As stated above we have in mind that the initial data of Equation (3.6), Ψ_0, will be already close to Φ. For example, Ψ_0 could be one of the approximate inertial manifolds studied in [17], [23], [43], [54], [57]. For many of these AIMs one shows that a thin neighborhood of these manifolds is invariant by obtaining an estimate for solutions near the manifold or on the attractor of the form

$$|A(q(t)-\Psi_0(p(t)))| \leqslant Error(\Psi_0), \tag{4.3}$$

where $u(t)=p(t)+q(t)$ and solves (2.1). The $Error(\Psi_0)$ is *typically* bounded by $C\lambda_{M+1}^{-r}$ for some rational number $r>0$. It is possible for the error to decrease exponentially in the dimension of the manifold. This occurs for example when the solutions of (2.1) have Gevrey class regularity (see [13], [27], [38]).

Notice that this immediately implies the estimate

$$|A(\Psi_0(p)-\Phi(p))| \leqslant Error(\Psi_0) \tag{4.4}$$

for all p such that $p+\Phi(p)$ is on the attractor. Indeed, for any $t_0>0$ arbitrary, and any $p\in PH$ from Lemma 3.1 there exists $u_0=p_0+\Phi(p_0)$ such that $p=PS(t_0)u_0$. Furthermore, $u(t)=S(t)u_0$ is on the attractor. Hence,

$$|A(\Phi(p(t))-\Psi_0(p(t)))| = |A(q(t)-\Psi_0(p(t)))| \leqslant Error(\Psi_0), \text{ for all } \quad t\geqslant 0,$$

where $q(t)=QS(t)u_0$. However, at $t=t_0$ Equation (4.4) follows. It is possible to prove (4.4) for solutions away from the attractor, but on the IM, since many of the necessary estimates hold on the IM as well. However, to do this would require us to look at a specific choice for Ψ_0.

Remark. In the case that the forcing term f in (2.1) is smooth, say $f\in\mathscr{D}(A^\gamma)$ for $\gamma>0$, then one can in general show that the solutions of the type of equations under consideration are bounded in $\mathscr{D}(A^{1+\gamma})$ for all $t>0$. As a consequence this implies that $|Aq(t)| \leqslant C/\lambda_{M+1}^\gamma$ for time sufficiently large. As the solutions become more smooth, that is, large γ, the q part of the solution becomes smaller. Moreover, the approximate inertial manifolds mentioned above also decay (in Fourier space) at least like $\lambda_{M+1}^{-\gamma}$ for this case. Thus $|A(q(t)-\Psi_0(p(t)))| \leqslant C\lambda_{M+1}^{-\gamma}$ for solutions inside the absorbing ball (see for example, [36], [38], [41]). Hence, by the above argument one can obtain over the support of Φ the trivial estimate $|A(\Psi_0(p)-\Phi(p))| \leqslant C\lambda_{M+1}^{-\gamma}$ for all of the AIMs mentioned above (including $\Psi_0=0$).

Remark. In the case $\gamma \gg 1$ and the solutions are smooth, $q(t)$ decays rapidly in Fourier space, there may be little advantage in approximating such q from a numerical point of view. This point is studied and demonstrated in detail in [13], [27], [36], [38].

In general we set

$$\sup_{p \in PH} |A(\Psi_0(p)-\Phi(p))| = \max_{|Ap| \leqslant 4M_2} |A(\Psi_0(p)-\Phi(p))| = Error(\Psi_0). \tag{4.5}$$

The following theorem gives the short time convergence.

THEOREM 4.1. *Let $\tau_0 = 1/\lambda_M$. Further, suppose that M is chosen so that* (2.5) *and* (4.12) (*below*) *are satisfied. Then for $n\tau_0 \leqslant \tau \leqslant (n+1)\tau_0$, $n=0, 1, 2, \ldots$ we have*

$$\sup_{p \in PH} |A(\Psi(\tau, p)-\Phi(p))| \leqslant K_{5,M}^{n+1}\, Error(\Psi_0)\, e^{-\lambda_{M+1}\tau/2}. \tag{4.6}$$

The constant $K_{5,M}$ depends on M. Moreover, $K_{5,M} \leqslant 2$ and tends to one as M goes to infinity.

Proof. As before we choose $u_{0a} \in \mathrm{Graph}(\Psi_0)$, $u_{0m} \in \mathrm{Graph}(\Phi)$ such that $PS(\tau_0)u_{0a} = p = PS(\tau_0)u_{0m}$. As in the estimates (4.1), (4.2), we have

$$|A(\Psi(\tau_0, p)-\Phi(p))| \leqslant |A(q_{0a}-q_{0m})|\, e^{-\lambda_{M+1}\tau_0/2}, \tag{4.7}$$

where $q_{0a} = Qu_{0a}$, $q_{0m} = Qu_{0m}$. We must estimate the right-hand-side of this inequality. Notice

$$\begin{aligned} |A(q_{0a}-q_{0m})| &\leqslant |A(\Psi_0(p_{0a})-\Phi(p_{0a}))| + |A(\Phi(p_{0a})-\Phi(p_{0m}))| \\ &\leqslant Error(\Psi_0) + \gamma\, |A(p_{0a}-p_{0m})|, \end{aligned} \tag{4.8}$$

where $p_{0a} = Pu_{0a}$, $p_{0m} = Pu_{0m}$. We set $\delta(t) = P(S(t)u_{0a} - S(t)\,u_{0m}) = p_a(t) - p_m(t)$. Then $\delta(\tau_0) = 0$ and $\delta(t)$ solves

$$\begin{aligned} &\frac{d\delta}{dt} + A\delta + P[F(p_a+\Phi(p_a)) - F(p_m+\Phi(p_m))] \\ &\qquad + P[F(p_a+q_a) - F(p_a+\Phi(p_a))] = 0. \end{aligned}$$

From their definitions $q_a(t) - \Phi(p_a(t)) = \Psi(t, p_a(t)) - \Phi(p_a(t))$. If we take the inner product with $-A^2\delta$, we obtain using Equations (2.4), (2.10), (4.2)

$$-\frac{1}{2}\frac{d\,|A\,\delta(t)|^2}{dt} \leqslant |A^{3/2}\delta|^2 + K_2(1+\gamma)\,|A\delta|\,|A^{1+\beta}\delta| + K_2 2K_3 e^{-\lambda_{M+1}t/2}|A^{1+\beta}\delta|$$
$$\leqslant (\lambda_M + K_2(1+\gamma)\lambda_M^\beta)\,|A\delta|^2 + K_2 2K_3 \lambda_M^\beta\, e^{-\lambda_{M+1}t/2}\,|A\delta|. \tag{4.9}$$

From the assumptions of Section 2 $|Ap_a(t)|$, $|Ap_m(t)|$ are continuous in time. Moreover, we may assume that $|A\,\delta(0)| \neq 0$, for otherwise there is nothing to prove. Thus there is some positive interval of time such that $|A\,\delta(t)| > 0$. Since the estimate we derive on $|A\,\delta(t)|$ is growing exponentially in time, we may assume without loss of generality that $|A\,\delta(t)| \neq 0$ on $0 \leqslant t < \tau_0$. We therefore devidide (4.9) by $|A\delta|$, apply Gronwall's lemma on the interval $[0, \tau_0]$ and use the fact that $|A\,\delta(\tau_0)| = 0$ to obtain

$$|A\,\delta(t)| \leqslant \frac{K_2 2K_3 \lambda_M^\beta}{\lambda_M + K_2(1+\gamma)\,\lambda_M^\beta}\,(\exp(\Lambda_M \tau_0) - \exp(\Lambda_M t)), \tag{4.10}$$

where $\Lambda_M = \lambda_M + K_2(1+\gamma)\ \lambda_M^\beta$ and where we have used the estimate $e^{-\lambda_{M+1}t/2} \leqslant 1$ in (4.9). We obtain using the mean value theorem on the exponentials and our choice for τ_0 that

$$|A\,\delta(0)| \leqslant K_4 \frac{2K_3}{\lambda_M^{1-\beta}},$$

where $K_4 = 2K_2 \exp(1 + K_2(1+\gamma)\,\lambda_M^{\beta-1})$.

Thus we see from (4.8) that

$$|A(q_{0a} - q_{0m})| \leqslant Error(\Psi_0) + \frac{\gamma K_4 2K_3}{\lambda_M^{1-\beta}},$$

and the estimate (4.2) can be improved. We have that

$$|A(\Psi(\tau, p) - \Phi(p))| \leqslant \left(Error(\Psi_0) + \frac{\gamma K_4 (2K_3)}{\lambda_M^{1-\beta}} \right) e^{-\lambda_{M+1}\tau/2}$$

for all $0 \leqslant \tau \leqslant \tau_0$ and for all $p \in PH$. Now we may repeat the above argument, each time improving the estimate (4.2), as many times as we like. If we do it n times, we find

$$|A(q_{0a} - q_{0m})| \leqslant Error(\Psi_0) + \frac{\gamma K_4 2K_3}{\lambda_M^{1-\beta}}\,Error(\Psi_0)$$
$$+ \left(\frac{\gamma K_4 2K_3}{\lambda_M^{1-\beta}}\right)^2 Error(\Psi_0) + \cdots + \left(\frac{\gamma K_4 2K_3}{\lambda_M^{1-\beta}}\right)^n 2K_3. \tag{4.11}$$

Now we require M large enough so that

$$\frac{\gamma K_4 2K_3}{\lambda_M^{1-\beta}} \leqslant \frac{1}{2}. \tag{4.12}$$

Since n is arbitrary, we must have that

$$|A(\psi(\tau, p) - \Phi(p))| \leqslant K_{5,\,M}\, Error(\Psi_0)\, e^{-\lambda_{M+1}\tau/2},$$

where

$$K_{5,\,M} := \frac{1}{1 - \gamma K_4 2K_3/\lambda_M^{1-\beta}} \leqslant 2.$$

For τ in the interval $[\tau_0,\ 2\tau_0]$ etc..., we can repeat the argument only with $\Psi_0(p) = \Psi(\tau_0, p)$, $\Psi(2\tau_0, p) \cdots$. ∎

Theorem 4.1 could be proven continuously in time without the above iteration procedure; however, the method here is suggestive of a time discretization. Indeed we have

Corollary 4.2. *For $\tau = \tau_0 = 1/\lambda_M$, we have that*

$$\sup_{p \in PH} |A(\Psi(\tau, p) - \Phi(p))| \leqslant K_{5,\,M} e^{-1/2} \frac{Error(\Psi_0)}{\lambda_M^{1-\beta}}.$$

Thus if one can integrate Equation (3.6) for the short time τ_0, one can improve the error of the initial guess Ψ_0 by the factor $K_{5,\,M} e^{-1/2} \lambda_M^{\beta-1} \leqslant 2\lambda_M^{\beta-1}$ (recall from Section 2 that $0 \leqslant \beta \leqslant 1/2$).

Before turning to the C^1 convergence we need two important auxiliary lemmas. In order to obtain the needed estimates to show C^1 convergence we will find it necessary to fix the interval of time we estimate the trajectories of the solutions to be independent of M. We chose for the rest of the paper the parameter τ_0 to be unity.

Lemma 4.3. *Suppose that $u_{0a} \in \mathrm{Graph}(\Psi((n-1)\tau_0))$ and $u_{0m} \in \mathrm{Graph}(\Phi)$ are such that $p = PS(\tau_0)\, u_{0a} = PS(\tau_0)\, u_{0m}$ with $\tau_0 = 1$. Then with $u_a(t) = S(t)\, u_{0a}$ and $u_m(t) = S(t)\, u_{0m}$ we have*

$$|A(u_a(t) - u_m(t))| \leqslant K_{8,\,M} e^{-(n-1)\,\lambda_{M+1}/2},$$

for all $0 \leqslant t \leqslant \tau_0$, $p \in PH$ and for all $n \geqslant 1$. The constant $K_{8,\,M}$ depends on M and is given by (4.14) *below.*

Proof. Notice that

$$q_a(t) := QS(t)\, u_{0a} = \Psi((n-1)\,\tau_0 + t,\ p_a(t)),\ q_m(t) := QS(t)\, u_{0m} = \Phi(p_m(t))$$

with $p_a(t)=PS(t)\,u_{0a}$ and $p_m(t)=PS(t)\,u_{0m}$. Moreover, the solutions u_a, u_m are not in the cone for $0\leqslant t\leqslant \tau_0$ since $p=PS(\tau_0)\,u_{0a}=PS(\tau_0)\,u_{0m}$ and $QS(\tau_0)\,u_{0a}=\Psi(n\tau_0,p)\neq QS(\tau_0)\,u_{0m}=\Phi(p)$. Thus we may obtain using Theorem 4.1

$$\begin{aligned}|A(u_a(t)-u_m(t))| &\leqslant (\gamma^{-1}+1)\,|A[\Psi((n-1)\,\tau_0+t,p_a(t))-\Phi(p_m(t))]|\\ &\leqslant (\gamma^{-1}+1)\{|A[\Psi((n-1)\tau_0+t,p_a(t))-\Phi(p_a(t))]|\\ &\quad+|A(\Phi(p_a(t))-\Phi(p_m(t)))|\}\\ &\leqslant (\gamma^{-1}+1)\{K_{5,M}^{k+1}\,Error(\Psi((n-1)\tau_0))\\ &\quad+\gamma\,|A(p_a(t)-p_m(t))|\},\end{aligned} \tag{4.13}$$

where since $\tau_0=1$, we must choose $k=[\lambda_M]+1$, ($\tau_0=\lambda_M^{-1}$ in Theorem 4.1).

The term $|A\,\delta(t)|=|A(p_a(t)-p_m(t))|$ was estimated in Theorem 4.1. However, we may use that theorem to improve the estimate. We proceed as in that theorem only instead of using (4.2) in (4.9), we use the conclusion of Theorem 4.1. That is, from Theorem 4.1

$$\begin{aligned}&|A[\Psi((n-1)\tau_0+t,p_a(t))-\Phi(p_a(t))]|\\ &\qquad\leqslant K_{5,M}^{k+1}\,Error(\Psi((n-1)\tau_0)\,e^{-\lambda_{M+1}t/2}.\end{aligned}$$

Using this estimate in place of (4.2) in (4.9), we find (4.10) becomes

$$|A\,\delta(0)|\leqslant\frac{K_2\lambda_M^{\beta}K_{5,M}^{k+1}\,Error(\Psi((n-1)\tau_0))}{\lambda_M+K_2(1+\gamma)\lambda_M^{\beta}}\,(\exp(\Lambda_M\tau_0)-\exp(\Lambda_M t)).$$

Applying the mean value theorem to the exponentials we conclude that

$$|A(p_a(t)-p_m(t))|\leqslant K_2\lambda_{M+1}^{\beta}\,K_{5,M}^{k+1}\,e^{\lambda_M+K_2(1+\gamma)\lambda_M^{\beta}}\,Error(\Psi((n-1)\tau_0)).$$

We have from (4.2) that $Error(\Psi((n-1)\tau_0))\leqslant 2K_3e^{-\lambda_{M+1}(n-1)\tau_0/2}$. The result follows after returning to (4.13) with

$$K_{8,M}:=(\gamma^{-1}+1)(K_{5,M}^{k+1}+\gamma 2K_2K_{5,M}^{k+1}\lambda_{M+1}^{\beta}2K_3e^{\lambda_M+K_2(1+\gamma)\lambda_M^{\beta}}), \tag{4.14}$$

where again $k=[\lambda_M]+1$. We also recall that $K_{5,M}$ in Theorem 4.6 satisfies $K_{5,M}\leqslant 2$. ∎

We recall that from Lemma 3.2 the operator $\rho_{a,op}$ defined by $PDu_a(t)$ with u_a defined as in the previous lemma, $u_a(t)=S(t)u_{0a}=p_a(t)+\Psi((n-1)\tau_0+t,\ p_a(t))$, and the differentiation is with respect to Pu_{0a} is invertible. That is, given any $\xi\in PH$, $\rho_a:=\rho_{a,op}\xi$ solves

$$\frac{d\rho_a}{dt}+A\rho_a+PF'(u_a(t))\mu_a=0 \tag{4.15}$$

with $\rho_a(0)=\xi$, $\mu_a(t)=\rho_a(t)+D\Psi((n-1)\tau_0+t,\ Pu_a(t))\,\rho_a(t)$ and $\rho_{a,\,op}$ is invertible for all $t>0$.

The next lemma was essentially proven in the proof of the cone condition in Theorem 2.2. Since the context is different here, we make the estimate more explicit.

LEMMA 4.4. *The operator $\rho_{a,\,op}$ satisfies*

$$\|A\rho_{a,\,op}^{-1}(t)\|_{L(PH,\,PH)}\leqslant \exp[(\lambda_M+K_2(1+\gamma)\,\lambda_M^\beta)\,t].$$

Proof. Taking the inner product of (4.15) with $-A^2\rho_a$ with $|A\xi|=1$ and using (2.3), we find

$$-\frac{1}{2}\frac{d\,|A\rho_a|^2}{dt}\leqslant \lambda_M|A\rho_a|+K_2\lambda_M^\beta\,|A\mu_a|.$$

The form for μ_a gives that $|A\mu_a|\leqslant(1+\gamma)|A\rho_a|$, where we recall that $\|A\,D\Psi\|_{L(PH,\,QH)}\leqslant\gamma$ from Lemma 3.2. Dividing by $|A\rho_a|$ ($\rho_{a,\,op}$ is invertible and $|A\xi|=1$, so $|A\rho_a|\neq 0$) applying Gronwall's lemma on the interval $[s,t]$, we find

$$|A\rho_a(s)|\leqslant|A\rho_a(t)|\,e^{(\lambda_M+K_2(1+\gamma)\,\lambda_M^\beta)(t-s)}.$$

Notice that at $s=0$, $|A\rho_a(0)|=|A\xi|$. Thus

$$|A\xi|\leqslant \exp[(\lambda_M+K_2(1+\gamma)\,\lambda_M^\beta)\,t]\,|A\rho_{a,op}(t)\,\xi|. \tag{4.16}$$

The result follows after taking the supremum over $|A\xi|=1$. ∎

We turn to convergence in the C^1 norm. We will require the nonlinear term to satisfy

$$|A^{1-\beta}(F'(u_1)-F'(u_2))\mu|\leqslant L_1\,|A(u_1-u_2)|\,|A\mu|. \tag{4.17}$$

We remark that we only assume (4.17) for convenience. In fact, it is enough to assume that F' is continuous. Since all of the physical equations mentioned in the introduction satisfy (4.17), in practice this assumption presents no real restriction.

THEOREM 4.5. *Let M be large enoug so that* (2.5), (4.12) *and* (4.23) (*below*) *hold. Then given $\varepsilon>0$ there exists a $T(\varepsilon)$ such that*

$$\sup_{p\in PH}\|A(D\Psi(\tau,p)-D\Phi(p))\|_{L(PH,\,QH)}\leqslant\varepsilon$$

for all $\tau\geqslant T(\varepsilon)$.

Proof. Let $\varepsilon > 0$ be given and $p \in PH$ be arbitrary. Let $\tau_0 = 1$. As in Lemma 4.3 for each $n \geqslant 1$ there exists $u_{0a} \in \text{Graph}(\Psi((n-1)\tau_0))$ and $u_{0m} \in \text{Graph}(\Phi)$ so that $p = PS(\tau_0)\, u_{0a} = PS(\tau_0)\, u_{0m}$ and $\Psi(n\tau_0, p) = QS(\tau_0)\, u_{0a}$, $\Phi(p) = QS(\tau_0)\, u_{0m}$, where $\tau_0 = 1$. We also set $p_{0a} = Pu_{0a}$ and $p_{0m} = Pu_{0m}$. We have that

$$q_a(t, p_{0a}) := QS(t)\, u_{0a} = \Psi((n-1)\,\tau_0 + t, PS(t)\, u_{0a})$$

$$q_m(t, p_{0m}) := QS(t)\, u_{0m} = \Phi(PS(t)\, u_{0m}),$$

for $t \geqslant 0$. By taking the derivative of q_a with respect to p_{0a} and q_m with respect to p_{0m} we obtain

$$\sigma_{a,\, op}(t) = D\Psi((n-1)\tau_0 + t, p_a(t))\, \rho_{a,\, op}(t),$$

$$\sigma_{m,\, op}(t) = D\Phi(p_m(t))\, \rho_{m,\, op}(t)$$

for all $0 \leqslant t \leqslant \tau_0$. Moreover, for $\xi \in PH$, $\mu_a = (\rho_{a,\, op} + \sigma_{a,\, op})\xi$, $\mu_m = (\rho_{m,\, op} + \sigma_{m,\, op})\xi$, are solutions to (2.11) with $S(t)u_0 = S(t)u_{0a}$ and $S(t)u_0 = S(t)u_{0m}$ respectively.

Notice that

$$\begin{aligned}[D\Psi((n-1)\tau_0 + t, p_a(t)) - D\Phi(p_m(t))]\, \rho_{a,\, op}(t)\xi \\ = (\sigma_{a,\, op}(t) - \sigma_{m,\, op})\xi + D\Phi(p_m(t))(\rho_{m,\, op}(t) - \rho_{a,\, op}(t))\xi.\end{aligned}$$

Hence, at time $t = \tau_0$, (2.10) together with the previous equation gives

$$|A(D\Psi(n\tau_0, p) - D\Phi(p))\, \rho_{a,\, op}(\tau_0)\xi| \leqslant (1+\gamma)\, |A(\mu_a(\tau_0) - \mu_m(\tau_0))|. \tag{4.18}$$

Thus we need an estimate on $|A(\mu_a(\tau_0) - \mu_m(\tau_0))|$.

We have that $\mu_a(t)$ is given by

$$\mu_a(t) = e^{-At}\mu_a(0) - \int_0^t e^{-(t-s)A} F'(S(t)u_{0a}(s))\, \mu_a(s)\, ds. \tag{4.19}$$

Also μ_m is given by (4.19) with a replaced with m. Using these expressions we obtain for the difference $\Delta\mu = \mu_a - \mu_m$

$$\begin{aligned}\Delta\mu(t) = e^{-At}\, \Delta\mu(0) - \int_0^t e^{-(t-s)A}[F'(u_a(s))\, \mu_a(s) - F'(u_m(s))\, \mu_a(s)]\, ds \\ - \int_0^t e^{-(t-s)A}[F'(u_m(s))\, \mu_a(s) - F'(u_m(s))\, \mu_m(s)]\, ds,\end{aligned}$$

where $u_a(t) = S(t)\, u_{0a}$, $u_m(t) = S(t)\, u_{0m}$.

One may obtain from Equation (2.11) the a priori estimate $|A\mu_a| \leqslant M_3$ on the interval $[0, \tau_0]$ for some $M_3 > 0$. Notice also that $\mu_{a,op}(0) = P + D\Psi((n-1)\tau_0, p_a(0))$ and $\mu_{m,op}(0) = P + D\Phi(p_m(0))$ and hence $\Delta\mu(0) \in QH$. Also we have the estimate $\|A^\beta e^{-tA}\|_{L(H,H)} \leqslant K_6 t^{-\beta}$ (see [29] for example). Hence, (4.17) and (2.4) imply that

$$|A\,\Delta\mu(t)| \leqslant e^{-\lambda_{M+1}t}\,|A\,\Delta\mu(0)| + \int_0^t \frac{K_6 L_1 M_3}{(t-s)^\beta}\,|A(u_a(s) - u_m(s))|\,ds$$

$$+ \int_0^t \frac{K_2 K_6}{(t-s)^\beta}\,|A(\mu_a(s) - \mu_m(s))|\,ds.$$

Applying Lemma 4.3 and Gronwall's lemma, [29] (p. 188), we conclude

$$|A\,\Delta\mu(\tau_0)| \leqslant K_7 e^{-\lambda_{M+1}\tau_0}\,|A\,\Delta\mu(0)| + K_{9,M}(\lambda_M)\,e^{-(n-1)\lambda_{M+1}/2}, \tag{4.20}$$

where $K_{9,M} = K_7 K_6 L_1 M_3 K_{8,M}(1-\beta)^{-1}$, $K_{8,M}$ is given by (4.14) and depends on M. Notice that although we used (4.17) to obtain (4.20), it is enough to only assume that F' is uniformly continuous.

We also have that $|A\,\Delta\mu(0)| = |A(\mu_a(0) - \mu_m(0))|$ satisfies

$$\begin{aligned}|A(\mu_a(0) - \mu_m(0))| &= |A(D\psi((n-1)\tau_0, p_a(0)) - D\Phi(p_m(0)))\xi| \\ &\leqslant [\,\|A(D\Psi((n-1)\tau_0, p_a(0)) - D\Phi(p_a(0)))\|_{L(PH,QH)} \\ &\quad + \|A(D\Phi(p_a(0)) - D\Phi(p_m(0)))\|_{L(PH,QH)}]\,|A\xi|.\end{aligned} \tag{4.21}$$

In order to obtain an estimate in the operator norm we will need to take the supremum over all $\xi \in PH$ such that $|A\rho_{a,op}(\tau_0)\xi| = 1$. Notice that since $\rho_{a,op}$ is invertible, all such vectors of unit length are obtained. Lemma 4.4, (4.16), provides the bound for such $|A\xi|$, namely,

$$|A\xi| \leqslant \exp(\lambda_M + K_2(1+\gamma)\,\lambda_M^\beta).$$

Thus for such ξ, (4.18) becomes

$$\begin{aligned}&|A(D\Psi(n\tau_0, p) - D\Phi(p))\,\rho_{a,op}(\tau_0)\xi| \\ &\quad\leqslant (1+\gamma)\,K_7\,e^{-\lambda_{M+1}} e^{\lambda_M + K_2(1+\gamma)\lambda_M^\beta}[\,\|A(D\Psi((n-1)\,\tau_0, p_a(0)) \\ &\qquad - D\Phi(p_a(0)))\|_{L(PH,QH)} + |A(D\Phi(p_a(0)) - D\Phi(p_m(0)))\|_{L(PH,QH)}] \\ &\qquad + (1+\gamma)\,K_{9,M} e^{-(n-1)\lambda_{M+1}/2}\end{aligned} \tag{4.22}$$

In view of the first term in this inequality we will need to require that

$$(1+\gamma)\,K_7 e^{-\lambda_{M+1}} e^{\lambda_M + K_2(1+\gamma)\,\lambda_M^\beta} \leqslant 1/2 \tag{4.23}$$

(any number less than one will do). From the gap condition we may require the exponent in (4.23) to be as negative as we like by increasing M if necessary. Recall that the constant K_7 in (4.23) arises from the use of Gronwall's inequality to conclude (4.20), and K_7 depends only on K_2, K_6, β, τ_0. Once (4.23) is satisfied, M is fixed once and for all.

We see from Lemma 4.3 $|AP(u_{0a}-u_{0m})|$ can be made arbitrarily small for all $p \in PH$ by requiring τ; (*i.e.* $n\tau_0$), sufficiently large. Also from [3], [8], [46], $D\Phi$ is continuous. It also has compact support, and hence is uniformly continuous. Thus for any $\varepsilon > 0$ there exists an $T_1(\varepsilon)$ such that the second term in (4.21) may be required to satisfy

$$\frac{1}{2}\,\|A(D\Phi(p_a(0)) - D\Phi(p_m(0)))\|_{L(PH,\,QH)} \leqslant \frac{\varepsilon}{8}$$

for all $p \in PH$ and $\tau \geqslant T_1(\varepsilon)$ with $p_a(0) = Pu_{0a}$ and $p_m(0) = Pu_{0m}$.

Returning to (4.22) and taking the supremum over all $\xi \in PH$ such that $|A\rho_{a,\,op}(\tau_0)\xi| = 1$, we conclude

$$\begin{aligned}\sup_{p\in PH}\,&\|A(D\Psi(n\tau_0, p) - D\Phi(p))\|_{L(PH,\,QH)}\\ &\leqslant \frac{1}{2}\sup_{p\in PH}\,\|A(D\Psi((n-1)\,\tau_0, p) - D\Phi(p))\|_{L(PH,\,QH)}\\ &\quad+\frac{\varepsilon}{8} + (1+\gamma)\,K_{9,\,M}e^{-(n-1)\,\lambda_{M+1/2}}.\end{aligned}$$

Choose n so large that

$$(1+\gamma)\,K_{9,\,M}e^{-(n-1)\,\lambda_{M+1/2}} \leqslant \frac{\varepsilon}{8}$$

for all $n \geqslant N_0(\varepsilon)$. Set $a_n = \sup_{p\in PH}\|A(D\Psi(n\tau_0, p) - D\Phi(p))\|_{L(PH,\,QH)}$. Then we have that

$$a_n \leqslant \frac{1}{2}\left(a_{n-1} + \frac{\varepsilon}{2}\right)$$

For such n we have iterating for $m \geqslant 0$

$$a_{n+m} \leqslant \frac{a_{n-1}}{2^{m+1}} + \frac{\varepsilon}{2}.$$

We require further $2\gamma 2^{-(m+1)} \leqslant \varepsilon/2$ (recall $|a_{n-1}| \leqslant 2\gamma$). Then the result follows for all $\tau \geqslant T(\varepsilon)$ with $T(\varepsilon) = \max\{T_1(\varepsilon), (N_0(\varepsilon)+m)\,\tau_0\}$. ∎

We remark that it is possible to obtain a rate of convergence in Theorem 4.5 under the assumption that Φ is C^2. However, at present to obtain this would require that the gaps in the spectrum of the operator A to be larger than what (2.5) requires. Unfortunately, none of the physical equations mentioned in the introductions satisfy this more restrictive gap condition.

5. C^1 Convergence of a Galerkin Method

In the previous two sections we constructed functions $\Psi(\tau, \cdot)$ that approximate Φ in the C^1 norm as τ is increased. In this section we show how to construct approximations to Φ that are finite-dimensional, but converge (in the C^1 sense) as the dimension is increased.

Consider the Galerkin approximation of (2.1) based on the eigenfunctions of the linear operator A given by

$$\frac{du_N}{dt} + Au_N + P_N F(u_N) = P_N f, \tag{5.1}$$

where P_N denotes the projection onto the span of the first N eigenfunctions of A, for $N > M$ and M is determined by Theorem 2.2 and (4.23). We will also assume from now on that $f \in \mathscr{D}(A^{1-\beta})$. The reason for this is that for the solutions of the Galerkin system (5.1) to approximate the solutions of (2.1) in $\mathscr{D}(A)$ one needs the solutions of (2.1) to be more regular than just in $\mathscr{D}(A)$. In particular, if $f \in \mathscr{D}(A^{1-\beta})$ using the variation of constants formula (see [29] for example) one can show that $u(t) \in \mathscr{D}(A^{2-\beta})$ for $t > 0$ for $u(t)$ solving (2.1) by obtaining an estimate on the time derivative. The assumptions on F are the same as in the previous sections.

As already noted in [22], [23] Equation (5.1) enjoys the same properties as (2.1). Indeed, the N eigenvalues of $P_N A$ are exactly the same as the first N eigenvalues of A. Thus for $N > M$ the spectral gap condition is satisfied for (5.1). Moreover, the constants in the proof of the IM for (5.1) can be chosen in a uniform way (independent of N) so that we have the existence of a function Φ_N: $PH \mapsto P_N QH$ such that the Graph(Φ_N) is an inertial manifold for Equation (5.1), (see [22], [23], [37] for details). Thus the equation

$$\frac{dp_N}{dt} + Ap_N + PF(p_N + \Phi_N(p_N)) = Pf, \tag{5.2}$$

where $p_N = P_{u_N}$, has the same long-time dynamics as (5.1) and remains of dimension M as $N \to \infty$. Further, we have from [22], [23], [37]

$$\sup_{p \in PH} |A(\Phi(p) - \Phi_N(p))| \leqslant \frac{K_{10}}{\lambda_N^{1-\beta}}. \tag{5.3}$$

This estimates requires that the forcing term f in (2.1) be in $\mathscr{D}(A^{1-\beta})$.

Equation (5.2) will play the role of the approximate inertial form

$$\frac{dp}{dt} + Ap + PF(p + \Phi_{app}(p)) = Pf. \tag{5.4}$$

Here we will show that $\Phi_N \to \Phi$ in the C^1 norm as $N \to \infty$. Thus Equation (5.2) will be a small C^1 perturbation of the inertial form for the PDE, Equation (1.4) as N increases. Since the long-time dynamics of the inertial form is the same as that of the original PDE, and the long-time dynamics of (5.2) is the same as that of (5.1), we see that the hyperbolic structures of the attractor studied in [48], for example, are preserved by the Galerkin scheme (5.1) for N sufficiently large (for a direct approach to the persistence of hyperbolic structures of the attractor for the Navier–Stokes equation approximated by the Galerkin scheme (5.1), see [5], [55], [58]). A similar result is proven for finite difference approximations in [32] and our approach is the same as the one given there. In particular, we have

THEOREM 5.1. *Let Φ, Φ_N be as described above with $f \in \mathscr{D}(A^{1-\beta})$. Further, suppose that M so that* (2.5), (4.12), *and* (4.23) *hold. Then given $\varepsilon > 0$ there exists a $N(\varepsilon) > M$ such that in addition to* (5.3) *we have that*

$$\sup_{p \in PH} \| A(D\Phi_N(p) - D\Phi(p)) \|_{L(PH, QH)} \leqslant \varepsilon \tag{5.5}$$

for all $N \geqslant M(\varepsilon)$.

Notice that this result also gives us a way to approximate the function $\Psi(\tau, p)$, and hence Φ. Because the function Ψ constructed in the previous sections is an infinite-dimensional vector-valued function, one would need to approximate Ψ in order to implemented it in a numerical scheme. Since the system (5.1) and its associated inertial manifold Φ_N satisfy the same estimates as the true inertial manifold, the previous two sections apply to the solution $\Psi_N(\tau, p)$ solving

$$\begin{gathered} \frac{\partial \Psi_N(\tau, p)}{\partial \tau} + A\Psi_N(\tau, p) - D\Psi_N(\tau, p)(Ap + PF(p + \Psi_N(\tau, p))) \\ + P_N QF(p + \Psi_N(\tau, p)) = 0 \\ \Psi_N(0, p) = P_N \Psi_0(p), \end{gathered} \tag{5.6}$$

where Ψ_0 satisfies (3.7) and we have again incorporated the forcing term f into the term F. In other words one may apply the results of the previous sections to the Galerkin system (5.1) (recall we had to first prove te cone condition for the Galerkin system in Theorem 2.2 before obtaining it for

(2.1)). We may conclude therefore that $\Psi_N(\tau, \cdot)$ approximates Φ_N in the C^1 sense as τ is increased. Together with the previous theorem we may conclude

Theorem 5.2. *Let Ψ_N solve* (5.6) *with $f \in \mathscr{D}(A^{1-\beta})$. Suppose that M is chosen so that* (2.5), (4.12), (4.23) *are satisfied. Then given $\varepsilon > 0$ there exists an $N(\varepsilon) > M$ and a $T(\varepsilon) > 0$ such that*

$$\sup_{p \in PH} [|A(\Psi_N(\tau, p) - \Phi(p))| + \|A(D\Psi_N(\tau, p) - D\Phi(p))\|_{L(PH, QH)}] \leqslant \varepsilon$$

for all $N \geqslant N(\varepsilon)$ and $\tau \geqslant T(\varepsilon)$.

One possible advantage of approximating Φ with Ψ_N is that implementing it in (5.4) yields a system of dimension M that may be chosen to be close to the true inertial form in the C^1 norm.

Proof of Theorem 5.1. The proof follows the proof of Theorem 4.5 will little modification. Essentially one replaces $\Psi(n\tau_0, p)$ in that proof with $\Phi_N(p)$. Moreover, (5.3) will take the place of (4.2). There are some minor differences in the details.

Let $\varepsilon > 0$ be given and $p \in PH$ be arbitrary. Since Graph(Φ_N) and Graph(Φ) are inertial manifolds (and in particular invariant manifolds) for (5.1) and (2.1) respectively, we have that for every $p \in PH$ there exists $u_{0a} \in \text{Graph}(\Phi_N)$ and $u_{0m} \in \text{Graph}(\Phi)$ such that $PS_N(\tau_0) u_{0a} = p = PS(\tau_0) u_{0m}$, where $\tau_0 = 1$ and $S_N(t)$ is the semigroup for (5.1).

Following the proof of Theorem 4.1 (in particular, Equations (4.9), (4.10)) we have using (5.3) that for $\delta(t) = P(S_N(t) u_{0a} - S(t) u_{0m}) = p_N(t) - p_m(t)$

$$|A\delta(t)| \leqslant \frac{K_{11, M}}{\lambda_N^{1-\beta}}, \tag{5.7}$$

where $K_{11, M} = K_2 K_{10} \lambda_M^\beta \exp(\lambda_M + (1+\gamma)\, \lambda_M^\beta)$. In addition,

$$\begin{aligned} |A(u_m(t) - u_N(t))| &\leqslant |A\, \delta(t)| + |A(\Phi_N(p_N(t)) - \Phi(p_m(t)))| \\ &\leqslant \frac{K_{12, M}}{\lambda_N^{1-\beta}}, \end{aligned} \tag{5.8}$$

where $K_{12, M} = (1+\gamma)\, K_{11, M} + K_{10}$ and depends on M. This last estimates plays the role of Lemma 4.3 in the proof of Theorem 4.5. It can also be viewed as giving the continuous convergence of the semigroups $S_N(t)$ and $S(t)$ for finite intervals of time.

As before we will find it necessary to study the linearizations of the evolution equations. Taking the derivative with respect to the initial data of the solution $u_N(t)$ solving (5.1) we have that $\mu_N(t) := S'(t, u_{0a})\mu_{0,N}$ solves

$$\frac{d\mu_N}{dt} + A\mu_N + P_N F'(u_N(t))\mu_N = 0, \qquad \mu(0) = \mu_{0,N} \tag{5.9}$$

with $\mu_{0,N} \in P_N H$. As in the proof of Theorem 4.5 we will need to show that μ_N approximates μ_m with μ_m defined as in Theorem 4.5. This will show that the semigroups $S_N(t)$ and $S(t)$ approximate one another in the C^1 sense over finite time intervals.

One may obtain the estimate the time derivative of μ_m (see Lemma 3.5.1 of [29] or p.114 of [47])

$$\left| A^{1-\beta} \frac{d\mu_m}{dt} \right| \leqslant \frac{K_{13}}{t^{1-\beta}}.$$

Since $A\mu_m = -d\mu_m/dt - F'(u)\mu_m \in \mathscr{D}(A^{1-\beta})$, and $\lambda_N^{1-\beta} |AP_N Q\mu_m| \leqslant |A^{2-\beta}\mu_m|$ we have that

$$|A^{2-\beta} \mu_m(t)| \leqslant \frac{K_{13}}{t^{1-\beta}}, \quad |AP_N Q\mu_m(t)| \leqslant \frac{K_{13}}{t^{1-\beta}\lambda_N^{1-\beta}} \tag{5.10}$$

for all $0 < t \leqslant \tau_0$. Following the calculations leading to (4.20) only using (5.8)

$$\begin{aligned} |AP_N(\mu_N(t) - \mu_m(t))| \leqslant{}& K_6 e^{-\lambda_{M+1}t} |A(\mu_N(0) - \mu_m(0))| \frac{K_{12,M}}{\lambda_N^{1-\beta}} \\ &+ \int_0^t \frac{K_2 K_6}{(t-s)^\beta} |A(\mu(s) - \mu_N(s))| \, ds, \end{aligned}$$

where we have used the fact that as in Theorem 4.5 $(\mu_N(0) - \mu_m(0)) \in QH$. Also recall that the constant K_6 comes from the estimate $\|A^\beta e^{-tA}\|_{L(H,H)} \leqslant K_6 t^{-\beta}$. Therefore, using (5.10)

$$\begin{aligned} |A(\mu_N(t) - \mu_m(t))| &\leqslant |AP_N(\mu_N(t) - \mu_m(t))| + |P_N QA\mu_m(t)| \\ &\leqslant K_6 e^{-\lambda_{M+1}} |A(\mu_N(0) - \mu_m(0))| + \frac{K_{12,M}}{\lambda_N^{1-\beta}} + \frac{K_{13}}{t^{1-\beta}\lambda_N^{1-\beta}} \\ &\quad + \int_0^t \frac{K_2 K_6}{(t-s)^\beta} |A(\mu(s) - \mu_N(s))| \, ds. \end{aligned}$$

We obtain from Gronwall's lemma ([29], p. 188) at $t=\tau_0$

$$|A(\mu_N(\tau_0)-\mu_m(\tau_0))| \leqslant K_7 e^{-\lambda_{M+1}} |A(\mu_N(0)-\mu_m(0))| + \frac{K_{14,\,M}}{\lambda_N^{1-\beta}},$$

where K_7 is the same as in Theorem 4.5 and $K_{14,\,M}=K_7(K_{12,\,M}+K_{13})$.

Since $u_m(t)$ and $u_N(t)$ are on their respective inertial manifolds, we have

$$q_m(t)=\Phi(p_m(t)), \quad q_N(t)=\Phi_N(p_N(t))$$

with $q_N(t)=Qu_N(t)$. Taking the derivative of these last two expressions with respect to $p_m(0)$ and $p_N(0)$ respectively, we find

$$\sigma_{m,\,op}(t)=D\Phi(p_m(t))\,\rho_{m,\,op}\,, \quad \sigma_{N,\,op}(t)=D\Phi_N(p_N(t))\,\rho_{N,\,op}(t),$$

where $\mu_m(t)=\mu_{m,\,op}(t)\xi=(\rho_{m,\,op}(t)+\sigma_{m,\,op}(t))\xi$ and $\mu_N(t)=\mu_{N,\,op}(t)\xi=(\rho_{N,\,op}(t)+\sigma_{N,\,op}(t))\xi$ solve Equations (2.11), (5.9) respectively, for $\xi\in PH$. Notice also that $\mu_{m,\,op}(0)=P+D\Phi(p_m(0))$ and $\mu_{N,\,op}(0)=P+D\Phi_N(p_N(0))$. Hence, $(\mu_{m,\,op}(0)-\mu_{N,\,op}(0))\xi\in QH$ for all $\xi\in PH$.

We have

$$\begin{aligned}(D\Phi(p_m(t))-D\Phi_N(p_N(t)))\,\rho_{m,\,op}(t)\xi\\ =(\sigma_{m,\,op}(t)-\sigma_{N,\,op})\xi+D\Phi_N(p_N(t))(\rho_{N,\,op}(t)-\rho_{m,\,op}(t))\xi.\end{aligned}$$

Hence, at time $\tau_0=1$

$$\begin{aligned}&|A(D\Phi(p)\,\rho_{m,\,op}(\tau_0)-D\Phi_N(p)\,\rho_{m,\,op}(\tau_0))\xi|\\ &\quad\leqslant(1+\gamma)\,|A(\mu_m(\tau_0)-\mu_N(\tau_0))|,\end{aligned} \tag{5.11}$$

where we are using the fact that like $D\Phi$, $D\Phi_N$ satisfies $\|AD\Phi_N(p)\|_{L(PH,\,QH)}\leqslant\gamma$ for all $p\in PH$, $N>M$. Notice that

$$\begin{aligned}|A(\mu_m(0)-\mu_N(0))| \leqslant [\,&\|A(D\Phi(p_m(0))-D\Phi(p_N(0)))\|_{L(PH,\,QH)}\\ &+\|A(D\Phi(p_N(0))-D\Phi_N(p_N(0)))\|_{L(PH,\,QH)}]\,|A\xi|.\end{aligned}$$

Thanks to (5.7) we may obtain due to the uniform continuity of D_Φ (as in Theorem 4.5) that $(1/2)\,\|A(D\Phi(p_m(0))-D\Phi(p_N(0)))\|_{L(PH,\,QH)}\leqslant\varepsilon/4$ for $N\geqslant N_1(\varepsilon)$ for $N_1(\varepsilon)$ sufficiently large.

Precisely as in Theorem 4.5, in particular with the use of (4.23), we find that taking the supremum over all $\xi\in PH$ such that $|A\rho_{m,\,op}(\tau_0)\xi|=1$ (one may obtain the necessary bound on $\rho_{m,\,op}^{-1}$ as in Lemma 4.4) that

$$\sup_{p \in PH} \|A(D\Phi(p) - D\Phi_N(p)\|_{L(PH,\,QH)}$$

$$\leqslant \frac{\varepsilon}{2} + \frac{1}{2} \sup_{p \in PH} \|A(D\Phi(p) - D\Phi_N(p)\|_{L(PH,\,QH)},$$

where we require that for $N \geqslant N_2(\varepsilon)$, $(1+\gamma)\, K_{14,\,M}/\lambda_N^{1-\beta} \leqslant \varepsilon/4$. The theorem follows with $N(\varepsilon) = \max\{N_1(\varepsilon), N_2(\varepsilon)\}$. ∎

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